Complexity Theory

Homework Sheet 2

(Turn in before the lecure of Monday 14 Apr.)

7 April, 2014

Exercise 1. Prove that if $\overline{SAT} \in$ NP, then co-NP = NP. Hint: show that if $A \leq_m^p B$, then $\overline{A} \leq_m^p \overline{B}$.

Exercise 2. A colouring of a graph with c colours is an assignment of a number from $1, 2, \ldots, c$ to each vertex such that no adjacent vertices get the same number.

(a) Show that the following problem is in P.

Two-colouring: 2COL = $\{G$: graph G has a colouring with two colours$\}$

(b) Contrary to the two-colour version, the three-colouring problem is NP-complete.

Three-colouring: 3COL = $\{G$: graph G has a colouring with three colours$\}$

Now consider the following scenario: A mysterious (but trustworthy) wizard gives you a graph, and promises you that it is colourable with three colours.

Is it possible to come up with an efficient algorithm that finds a three-colouring of a graph, if you already know that such a colouring exists?

Exercise 3. Let $S = \{x_1, \ldots, x_n\}$ be a set of n elements, and let $\mathcal{C} = \{C_1, \ldots, C_m\}$ be a family of subsets of S. We say that $(S, \mathcal{C})$ has a hitting set of size at most k if there exists a subset $S' \subseteq S$ such that every set in $\mathcal{C}$ contains at least one element of S', and $|S'| \leq k$.

HITSET = $\{\langle S, \mathcal{C}, k\rangle : (S, \mathcal{C})$ has a hitting set of size at most $k\}$.

(a) Prove that HITSET is NP-complete by a reduction from 3SAT.

(b) Show that this problem is 2-query disjunctive length-decreasing self-reducible, i.e., that there are two polytime-computable length-decreasing[1] functions f_1, f_2 such that

$$\langle S, \mathcal{C}, k\rangle \in \text{HITSET} \iff f_1(\langle S, \mathcal{C}, k\rangle) \in \text{HITSET or } f_2(\langle S, \mathcal{C}, k\rangle) \in \text{HITSET}.$$

Hint: Consider an element $x \in S$. What if x is in the hitting set S'? What if it's not?

[1] A function $g : \{0,1\}^* \to \{0,1\}^*$ is called *length-decreasing* if the length of its output is strictly less than the length of its input.

(c) Show that search reduces to decision for HITSET. That is, assuming you have a procedure to decide HITSET in a single step, devise a polynomial-time algorithm which will, when given S, $\mathcal{C}$, and k, output a set $S' \subseteq S$ that is a hitting set for $\mathcal{C}$ of size at most k, if there is one, and rejects if there exists no hitting set that is small enough. Do this directly by using your previous self-reduction.

Exercise 4. **(a)** Show that there is no maximum set for $\leq_m^p$ reductions, i.e., that for any set A there is a set $B \not\leq_m^p A$ (Hint: is there an enumeration of all possible $\leq_m^p$-reductions?).

(b) Show that there is no maximal set, i.e., that for any set A there is a set B such that $A \leq_m^p B$ and $B \not\leq_m^p A$.

Exercise 5. Recall the non-deterministic time-hierarchy theorem. Now show that NTIME$(n) \neq$ P (Hint: use padding to show that NTIME$(n) \subseteq$ P implies NP $\subseteq$ P).[2] Note that although we ask you to show that these two classes differ, we don't ask you if either class is contained in the other, and we ourselves don't know!

[2]NTIME(n) can be defined as the class of sets decidable through linear-size certificates verified in linear-time. So $A \in$ NTIME(n) iff there is a linear-time machine M and some constant c such that $x \in A \iff \exists u \in \{0,1\}^{c|x|} M(x,u) = 1$.